“Cards, Dice and Distributions”

Introductory activities for the Binomial and Poisson distributions using cards and dice.
Using the Probability distribution of the throw of a dice and the draw of a card to provide a visual path through Expectation of functions of random variables and linear combinations of independent random variables.

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Introduction to the Binomial Distribution

Activity.

Use the picture cards from a deck of cards, four kings, four queens and four jacks.

Tell the students you are going to choose one of the cards without looking, at random. They need to write down their guess for the type of card you will draw, a king or a queen or a jack. Draw a card. Count how many people guessed correctly.

Questions "What is the probability of guessing correctly?"
"What is the chance you will guess wrong?"
"Are you more likely to be right or wrong?"
"Did the outcome of one trial affect the outcome of the next one?"

The Test: The students write 1 to 8 and, for each, their guess for the outcome of that trial.

Draw a card, the students "mark" the first question, you replace the card and draw again a total of 8 times. Each student counts their number of correct guesses.

The variable of interest is X, the number of correct guesses out of eight.

Questions "What values X can take?"

Make a table on the board and collect data from class.

x	**frequency**	**proportion from experiment**	**theoretical probability**

Question "What should the total of the frequency column be? Is it correct?"
It is quite likely it will not be correct, either you missed a student's hand or someone didn't put their hand up.

"How do we calculate the proportion?"

In general a bigger sample gives more reliable results so it would be a good idea to repeat the experiment a couple more times. It is quick and it gives the students time to start to think about what they are doing i.e. the conditions

Add the new data to the previous results.
Calculate the proportions.
Questions "Look at the results, do they look reasonable?"
"Which proportion do you think would be the greatest?" "Why"
"Why do you think the proportion for 8/8 is less than 0/8?"

Draw a bar graph of this data.

On to the Theory
This was an experimental situation. Now we can also calculate the theoretical probabilities.

Start with the easier ones. Encourage them to work in pairs.

Suggested starting place: What is the chance/probability of getting all the answers correct in a particular "test" (set of eight questions)? This is $P(X = 8)$
Reminders Multiplication principle
All events are independent.
You may need to draw a probability tree to nudge their memories. Fill in calculation and answer on table.
Students can now calculate P(X=0)

Students calculate P(X = 1).
Ask the students who got exactly one answer correct in one of the "tests" to tell you which question they got correct. Calculate probability of this particular result occurring, e.g. the probability of getting the fourth guess correct and all the others wrong.

Students now calculate **P(X = 1).**
Students who "see" probability trees easily can see the eight possible branches.
Reminders How many ways to choose 1 from 8.
Test could look like √ x x x x x x x or x x x x √ x x x

<u>It is worth taking time for students to get this bit as it is the key to their getting to a *general formula* by themselves.</u>

Now P(X= 7) should be easy.

The last conceptual leap is getting P(X = 2) where they need to use the idea of choosing 2 from 8 to get a value of 28 to multiply $(\frac{1}{3})^2(\frac{2}{3})^6$ by.
From here on it should be reasonably plain sailing as **they calculate all the theoretical probabilities using the ideas already developed.**

Draw a bar graph of the theoretical probabilities. Compare it with the proportions from the experimental data. Hopefully they look sufficiently similar to be convincing.

Finally "go general" by going through another particular example,
"What if there had been four types of card to choose from and you had to guess 9 times in each "test"? Calculate the probability of getting exactly 2 correct."
"What if the probability of success was **p** and we had **n** trials?"

Tie up with discussion of conditions
F fixed number of trials
I independent events
S probability of success constant
T two outcomes
referring back to the particular situation.

<u>An experiment to illustrate a situation which can be modeled using the Poisson Distribution.</u>

To the best of my knowledge this adaptation of the hokey pokey experiment was developed by Tim Leadbeater, a Diploma in Teaching(Secondary) student at the University of Auckland.

The random variable, Y, in this experiment is the number of double sixes thrown in a period of three minutes.

Each student gets two dice. They roll them again and again for three minutes. They keep a record of the times they throw a double six. They do not have to count the number of times they roll the dice. (Is this a problem?)

The data is collected on the board.

x	**Frequency**	**Relative Frequency**	**Theoretical Probability**

Questions. "Are the conditions the same as those for the experiment which was modeled by the binomial distribution?"
"What is the highest number of double sixes you could possibly get?" "What's the highest number we got?"
"How likely are you to get a double six on a single throw?"
"Can you throw two double sixes at the same time?"
"If you throw a double six this time are you more likely/less likely to throw one with the next throw?"
"Can you predict when you are going to throw a double six?"
"Are you twice as likely to get a double six in 10 seconds as you are in 5 seconds?" (Does this seem the right question?)

As we know that a bigger sample size brings the long run relative frequency closer to the theoretical probability once again it seems advisable to repeat the experiment to collect more data.

Students can calculate the proportions for each outcome. They also need to calculate the mean and the variance. Hopefully these will be close to each other in value.

Draw a bar graph of the data. (Look at the difference between it and the graph for the Binomial Distribution.)

Now let the students compute the theoretical probabilities using the formula.

$P(X=x) = \dfrac{e^{-\lambda}}{x!}\dfrac{\lambda^x}{1}$ and using the mean number of double sixes per three minutes as λ. Alternatively use the values in the table of Poisson probabilities if λ has a convenient value.

Draw a bar graph of this data and compare it with the experimental proportions.

Tie up with a discussion of the conditions which need to be satisfied for the Poisson to be an appropriate model for a distribution..